

Evolutionary Hamiltonian Graph Theory

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Abstract

The concept of NP -completeness was introduced in 1971 by Stephen Cook, who conjectured that NP -complete problems are not solvable in polynomial time. Today, this conjecture seems much more reasonable motivated by the fact that the developments arising around various NP -complete problems have undergone a natural gradual growth and evolution, generating a great diversity. These developments provide an exclusive valuable domain beyond biology with continuously growing diversity and well described environment-origins-gene structures relations (the latter cannot be said about living forms originated about 3.7 billion years ago with too slow changes) towards better understanding the universal mechanisms to explain evolution in a wide variety of domains outside of biology. In this paper we focus on one of the most heavily studied areas, that joins together a number of NP -complete cycle problems, called large cycles theory - a simplified version of well-known hamiltonian graph theory, to show that the theorems on this subject evolve and adapt to their environment generating a great diversity by an iterative process from simplicity to complexity by certain hereditary mechanisms.

Key words. Evolution, universal evolution, fundamental results, hamiltonian graph theory, large cycles theory.

1 Introduction

A Hamilton cycle of a graph is a cycle which passes through every vertex of the graph exactly once, and a graph is hamiltonian if it contains a Hamilton cycle. Classic hamiltonian problem; determining when a graph contains a Hamilton cycle, is one of the most central notions in graph theory and is one of the most attractive and most investigated problems among NP -complete problems that Karp listed in his seminal paper [40]. Cook [18] conjectured that one cannot hope for a simple classification of hamiltonian graphs. In other words, it seems to be impossible to obtain a criterion for a graph to be hamiltonian which implies

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a polynomial-time algorithm. This fact gave rise to a growing number of conditions that are either necessary or sufficient. Today, this conjecture seems much more reasonable motivated by the fact that the developments arising around various *NP*-complete problems have undergone a natural gradual growth and evolution, generating a great diversity. These developments provide an exclusive domain beyond biology with continuously growing diversity and well described environment-origins-gene structures relations toward better understanding the universal mechanisms to explain evolution in a wide variety of other domains, including psychology, economics, culture, language, medicine, computer science and physics.

In this paper we focus on one of the most heavily studied areas that joins together a number of *NP*-complete cycle problems, called large cycles theory - a simplified version of well-known hamiltonian graph theory. We show that the theorems on this subject evolve and adapt to their environment generating a great diversity. As expected, these theorems evolve by an iterative process from simplicity to complexity, from primitive beginnings (such as "every complete graph is hamiltonian") to best possible theorems.

This simplified and valuable model has a number of advantages with respect to biology and can be useful towards better understanding the universal mechanisms to explain evolution in a wide variety of domains outside of biology.

- Large cycles theory, originated about 60 years ago, evolves much more rapidly than living forms on Earth, originated about 3.7 billion years ago.
- The origins of theorems in large cycles theory can be strongly determined by exact branchings of the tree of developments.
- Genetic units and hereditary mechanisms in large cycles theory are much more simpler than gene structures of living forms.

At the most fundamental level, Charles Darwin's theory of evolution states that organisms evolve and adapt to their environment by an iterative process. This process can be conceived as an evolutionary algorithm that searches the space of possible forms for the ones that are best adapted. After those fit variants are retained, they can again undergo variation, either directly or in their offspring, starting a new round of the iteration. The overall mechanism is similar to the problem-solving procedures of trial-and-error or generate-and-test: evolution can be seen as searching for the best solution for the problem of how to survive and reproduce by generating new trials, testing how well they perform, eliminating the failures, and retaining the successes.

Donald Campbell was one of the first authors to formulate a generalized Darwinian algorithm directly applicable to phenomena outside of biology [15]. He wanted to build a science of science, which Campbell termed evolutionary epistemology. He characterized biology as the study of "progressive adaptation" and made an abstraction of the mechanism of natural selection by introducing the blind-variation-and-selective-retention scheme. Finally, he distinguishes ten different levels that are applicable to biological and social evolution.

Similar mechanisms are extensively applied in the domains of genetic algorithms and evolutionary computation, which develop solutions to complex problems via a process of variation and selection.

Universal Darwinism (also known as generalized Darwinism or universal selection theory) refers to a variety of approaches that extend the theory of Darwinism beyond its original domain of biological evolution on Earth. The idea is to formulate a generalized version of the mechanisms of variation, selection and heredity proposed by Charles Darwin, so that they can be applied to explain evolution in a wide variety of other domains, including psychology, economics, culture, language, medicine, computer science and physics. In result, a number of new areas have been appeared such as evolutionary psychology, evolutionary economics, evolutionary computation, evolutionary algorithm, evolutionary thought and so on.

In this paper we present an analogous evolutionary theory concerning mathematics. More precisely, we focus on one of the most heavily studied areas in graph theory, that joins together a number of *NP*-complete cycle problems, called large cycles theory - a simplified version of well-known hamiltonian graph theory, to explain the evolution mechanisms in this special subject in view of general evolutionary theory.

We deal a special attention to so called "fundamental theorems", by observing that all theorems in large cycles theory have descended from a number of common ancestors, called fundamental theorems (see Section 7). Remember, that the term "fundamental result" is used in various fields of science to characterize mainly the central and most important results in the area. The list of fundamental results in large cycles theory are presented in Section 8.

In the next section, we give necessary terminology and notations. Section 3 is devoted to complexity classes of computational problems, including *NP*-complete problems. The general environment (large cycles theory) and the patterns evolved in this environment (theorems in large cycles theory) are introduced in Section 4. Next, we describe the structure of patterns (theorems) based on a number of notions such as research objects (types of cycles that are investigated in large cycles theory), research tools (graph invariants, etc.), conditions and sharpness. Evolution mechanisms are described in Section 6, where four evolution levels in large cycles theory are distinguished including improvements (or strengthening), generalizations, advanced genetic combinations and modifications.

2 Terminology

Throughout this article we consider only finite undirected graphs without loops or multiple edges. A good reference for any undefined terms is [13]. Denote by $d(x)$ the degree of a vertex x in the graph G . The neighborhood of x will be denoted by $N(x)$.

A simple cycle (or just a cycle) C of length t is a sequence $v_1v_2...v_tv_1$ of distinct vertices $v_1, ..., v_t$ with $v_iv_{i+1} \in E(G)$ for each $i \in \{1, ..., t\}$, where $v_{t+1} =$

v_1 . When $t = 2$, the cycle $C = v_1v_2v_1$ on two vertices v_1, v_2 coincides with the edge v_1v_2 , and when $t = 1$, the cycle $C = v_1$ coincides with the vertex v_1 . So, by this standard definition, all vertices and edges in a graph can be considered as cycles of lengths 1 and 2, respectively. If Q is a cycle then we use $|Q|$ to denote the length of Q , that is $|Q| = |V(Q)|$. A path (cycle) on n vertices is denoted by P_n (C_n , respectively).

A graph G is hamiltonian if G contains a Hamilton cycle, i.e. a cycle of length n . We call Q a large cycle in a graph G if it dominates some certain subgraph structures in G in a sense that every such structure has a vertex in common with Q . If Q dominates all vertices in G then clearly C is a Hamilton cycle. A cycle Q is a dominating cycle if it dominates all edges in G . A cycle Q is a PD_λ (path dominating) cycle if it dominates all paths in G of length at least some fixed integer λ . Finally, a cycle Q is a QD_λ (cycle dominating) cycle if it dominates all cycles in G of length at least λ .

We reserve n, q, δ, κ and α to denote the number of vertices (order), number of edges (size), minimum degree, connectivity and independence number of a graph, respectively. The length c of a longest cycle in a graph is called the circumference. For C a longest cycle in G , let \bar{p} and \bar{c} denote the lengths of a longest path and a longest cycle in $G \setminus C$, respectively. Let $s(G)$ denote the number of components of a graph G . A graph G is t -tough if $|S| \geq ts(G \setminus S)$ for every subset S of the vertex set $V(G)$ with $s(G \setminus S) > 1$. The toughness of G , denoted $\tau(G)$, is the maximum value of t for which G is t -tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). Woodall [58] defined the binding number $b(G)$ of a graph G as follows:

$$b(G) = \min_{X \in F} \frac{|N(X)|}{|X|},$$

where $F = \{X : \emptyset \neq X \subseteq V(G)\}$ and $N(X) = \cup_{x \in X} N(x)$.

A graph G is said to be planar if G is embeddable into the plane without crossing edges. A projective plane, sometimes called a twisted sphere, is a surface without boundary derived from a usual plane by addition of a line at infinity. Just as a straight line in projective geometry contains a single point at infinity at which the endpoints meet, a plane in projective geometry contains a single line at infinity at which the edges of the plane meet. A projective plane can be constructed by gluing both pairs of opposite edges of a rectangle together giving both pairs a half-twist. It is a one-sided surface, but cannot be realized in three-dimensional space without crossing itself.

A graph G is the intersection graph of subgraphs H_1, \dots, H_m of a graph H if the vertices of G one-to-one correspond to the subgraphs H_1, \dots, H_m and two vertices of G are adjacent if and only if the corresponding subgraphs intersect.

A graph is an interval graph if and only if it is an intersection graph of subpaths of a path. Next, a graph is a split graph if and only if it is an intersection graph of subtrees of a star, i.e., a graph $K_{1,m}$. Further, a graph is chordal if and only if it is an intersection graph of subtrees of a tree. Finally, a comparability graph is a graph whose edges can be transitively oriented (i.e. if $x > y$ and $y > z$, then $x > z$); a cocomparability graph G is a graph whose complement \bar{G}

is a comparability graph. Spider graphs are the intersection graphs of subtrees of subdivisions of stars. Thus, spider graphs are chordal graphs that form a common superclass of interval and split graphs.

If a graph G contains no induced copy of H , we say that G is H -free.

Let a, b, t, k be integers with $k \leq t$. We use $H(a, b, t, k)$ to denote the graph obtained from $tK_a + \overline{K}_t$ by taking any k vertices in subgraph \overline{K}_t and joining each of them to all vertices of K_b . Let L_δ be the graph obtained from $3K_\delta + K_1$ by taking one vertex in each of three copies of K_δ and joining them each to other. For odd $n \geq 15$, construct the graph G_n from $\overline{K}_{\frac{n-1}{2}} + K_\delta + K_{\frac{n+1}{2}-\delta}$, where $n/3 \leq \delta \leq (n-5)/2$, by joining every vertex in K_δ to all other vertices and by adding a matching between all vertices in $K_{\frac{n+1}{2}-\delta}$ and $(n+1)/2 - \delta$ vertices in $\overline{K}_{\frac{n-1}{2}}$. It is easily seen that G_n is 1-tough but not hamiltonian. A variation of the graph G_n , with K_δ replaced by \overline{K}_δ and $\delta = (n-5)/2$, will be denoted by G_n^* .

3 Complexity classes of computational problems

Computational complexity theory focuses on classifying computational problems according to their inherent difficulty, and relating those classes to each other. Graph theory and combinatorics focus on particular problems and their real difficulties.

Significant progress has been made in combinatorics and graph theory toward improving our understanding of the inherent difficulty in computational problems and what can be computed efficiently. Today, most problems of known interest have been classified as to whether they are polynomial-time solvable or NP -complete.

An algorithm is said to be polynomial time if its running time is upper bounded by a polynomial expression in the size of the input for the algorithm. Problems for which a polynomial time algorithm exists belong to the complexity class P , which is central in the field of computational complexity theory. Polynomial time is a synonym for "tractable", "feasible", "efficient", or "fast". The following problems are polynomial-time solvable: shortest path problem, minimum spanning tree problem, linear programming, matching, Eulerian cycle problem, network flow problem and so on.

An algorithm is deterministic if at each step there is only one choice for the next step given the values of the variables at that step. An algorithm is non-deterministic if there is a step that involves parallel processing. A problem is said to be in the class NP of problems if it can be solved by an algorithm which is non-deterministic and has a time complexity function which is polynomial. NP problems are recognized by the fact that their solutions can be checked for correctness by a deterministic polynomial time algorithm. Every problem in P is also in NP . The non-deterministic algorithm that can be used is "guess the answer". The guess can be checked in polynomial time by the algorithm which solves the problem. A famous and long standing open problem is whether

or not $P = NP$. There is a collection of problems with the property that any polynomial time deterministic algorithm which solves one of them can be converted to a polynomial time algorithm which solves any other one of them (they are said to be polynomially equivalent problems) and if such an algorithm existed for any one of them, then $P = NP$. These problems are called NP -hard problems. NP -hard problems may or may not be NP problems. Those that are NP are called NP -complete problems. An example of an NP -complete problem is the Traveling Salesman Problem.

Today, most of important developments in discrete mathematics are centered on various NP -complete problems in trying to find different "effective layers" or "effective subspaces" in structures of NP -complete problems. In fact, by Cook's conjecture [18], NP -complete problems can not be covered by such layers. Today, after intensive investigations, many NP -complete problems are like unbreakable rock fragments with numerous cuttings and bore-holes.

4 General environment and patterns

Large cycles theory is considered as a general environment and the theorems in large cycles theory as patterns.

Large cycles theory traces its origins to 1855. Irish physicist, astronomer and mathematician Sir William Rowan Hamilton (1805-1865) invented the "Icosian Calculus", a noncommutative algebra so called because it involved a planar embedding of the graph of a dodecahedron, which has 20 vertices. The system has two operations: L and R , standing for "left" and "right" respectively, the idea being that if one has just arrived at a vertex, one can choose to go left or right, with the value 1 being reserved for an expression which returns to one's point of origin. For example, a path that turns right twice and then left once can be expressed as the term $R2L$. Similarly, since each face of a dodecahedron is pentagonal, we know that $R5 = L5 = 1$. Hamilton showed that symmetry notwithstanding, the equation

$$LLRRRLRLRLLRRRLRLR = 1$$

defines the only Hamiltonian Cycle on a dodecahedron. Since $LR \neq RL$, the Icosian Calculus is clearly noncommutative. However, it is associative. For example, $(LR)L = L(RL)$. Hamilton's first communication about his Icosian Calculus was to his friend Robert Graves in a letter dated Oct. 7th, 1856.

However, Hamilton Cycles should not have been named after Hamilton at all. In fairness, they should be called "Kirkman Cycles" after Thomas Penyngton Kirkman, the man who actually first discovered them. His interest in polyhedra led him to discover Hamilton cycles in a paper received by the Royal Society on Aug. 6th, 1855, predates Hamilton's earliest communication, let alone his first publication on the subject, by more than a year. However, precedence is not the only argument on Kirkman's side. Whereas Hamilton considered only the one special case of cycles in the dodecahedron, Kirkman's result was much more general, because he pondered the existence of Hamiltonian Cycles

in all graphs corresponding to planar embeddings of solid shapes. In addition, Kirkman was the first to discover an infinite class of non-hamiltonian polyhedra. He showed that any bipartite graph with an odd number of vertices must be non-hamiltonian. He gave an example of a planar, 3-connected, bipartite, non-hamiltonian graph.

If a graph G does not satisfy a sufficient condition for hamiltonicity, we cannot guarantee the existence of a Hamilton cycle. But if G is close to satisfy the condition, we may hope find some "hamiltonian-like" structures such as long cycles and hamiltonian paths. Further extensions of these notions lead to cycle and path covers, maximum matching, spanning trees with smallest number of leaves and many others that are rather far from their origins. Actually, each of these questions is really a part of the general area called "hamiltonian graph theory".

Large cycles theory can be considered as a simplified alternative to hamiltonian graph theory concerning the main "hamiltonian-like" structures in graphs. In fact, large cycles theory is a natural extension of classic hamiltonian problem including Hamilton cycles, longest cycles, dominating cycles, as well as some generalized cycles including Hamilton and dominating cycles as special cases. In the last 60 years, the developments in hamiltonian graph theory gave rise to a wide variety of results [8], [32], [33] concerning large cycles.

Systematic investigations of Hamilton cycles began only in 1952 when Swiss mathematician Gabriel Andrew Dirac (1925-1984) [21] discovered the first sufficient condition for the existence of a Hamilton cycle and the first lower bound for the length of a longest cycle in graphs, based on two simplest graph invariants - order n and minimum degree δ .

Theorem A (Dirac [21], 1952). Every graph with $\delta \geq \frac{n}{2}$ is hamiltonian.

Theorem B (Dirac [21], 1952). Every 2-connected graph either has a Hamilton cycle or has a cycle of length at least 2δ .

5 The structure of theorems

Informally, theorem is of the form of an indicative conditional:

$$\text{If } A \text{ then } B. \tag{1}$$

In this case, A is called the hypothesis (condition) of Theorem (1) and B the conclusion.

Conclusion B is related to possible types of large cycles. Large cycle structures are centered around well-known Hamilton (spanning) cycles. Other types of large cycles were introduced for different situations when the graph contains no Hamilton cycles or it is difficult to find it. Generally, a cycle C in a graph G is a large cycle if it dominates some certain subgraph structures in G in a sense that every such structure has a vertex in common with C . When C dominates

all vertices in G then C is a Hamilton cycle. When C dominates all edges in G then C is called a dominating cycle introduced by Nash-Williams [46]. Further, if C dominates all paths in G of length at least some fixed integer λ then C is a PD_λ (path dominating)-cycle introduced by Bondy [12]. Finally, if C dominates all cycles in G of length at least λ then C is a CD_λ (cycle dominating)-cycle, introduced in [50].

In (1), conclusion B usually appears in any of the following forms:

- (a1) G has a Hamilton cycle,
- (a2) G has a dominating cycle,
- (a3) every longest cycle in G is a dominating cycle,
- (a4) G has a CD_λ -cycle,
- (a5) every longest cycle in G is CD_λ -cycle,
- (a6) a lower bound for the circumference.

Sometimes, $B \equiv B_1 \vee B_2$ where

$$B_1 \in \{(a1), (a2), (a3), (a4), (a5)\}, \quad B_2 \equiv (a6).$$

As for hypothesis A , generally it can be presented as $A_1 \wedge A_2 \wedge \dots \wedge A_m$ where A_i for each $i \in \{1, \dots, m\}$, appears in the following forms:

- (b1) an algebraic (numerical) relation $f_1 \geq f_2$ between two algebraic expressions f_1, f_2 ,
- (b2) a structural limitation defined by forbidden subgraphs (examples: forbidden triangle, claw, P_6 , and so on),
- (b3) a structural limitation defined by direct description (examples: conditions for a graph to be regular, bipartite, interval, chordal, and so on).

The hypotheses and conclusions defined by $(a_1), (b_1), (b_2), (b_3)$, carry some genetic information (genome) in forms of initial graph invariants, generalized invariants, forbidden subgraphs and special graph classes (allele). There are a number of well-known basic (initial) invariants (allele) of a graph G occurring in various hamiltonian results and having significant impact on large cycle structures, namely order n , size q , minimum degree δ , connectivity κ , binding number $b(G)$, independence number α , toughness τ and the lengths of a longest path and longest cycle in $G \setminus C$ for a given longest cycle C , denoted by \overline{p} and \overline{c} , respectively.

Some of these basic gene elements, especially minimum degree δ , have been generalized in terms of degree sequences, degree sums, generalized degree, neighborhood unions and so on, giving rise many generalized theorems.

6 On sharpness

First of all, we need some additional definitions concerning relaxation and strengthening.

Definition 1. Let $f_1 \geq f_2$ be a condition in (1) defined by (b_1) . We say that $f_1 \geq f_2$ is can be relaxed in (1) if it can be replaced by $f_1 \geq f_2 - \epsilon$ for some positive ϵ .

Definition 2. Let " G is H_1 -free" be a condition in (1) defined by (b_2) . We say that " G is H_1 -free" is stronger than " G is H_2 -free" if H_1 is an induced subgraph of H_2 .

For example, " G is P_4 -free" is stronger than " G is P_5 -free" or " G is P_6 -free", and " G is $N_{0,0,0}$ -free" is stronger than " G is H -free" for each

$$H \in \{N_{0,0,1}, N_{0,0,2}, N_{0,1,1}, N_{0,0,3}, N_{0,1,2}, N_{1,1,1}\}.$$

If a theorem is not sharp (best possible, tight) then clearly it is incomplete and need further improvement.

Definition 3. A theorem is said to be sharp in all respects (partly, respectively) if its conclusion cannot be strengthened and each condition (some condition, respectively) in it cannot be relaxed under the same conclusion.

According to Definition 3, algebraic relations (see (b_1) in previous section) can be gradually (smoothly) relaxed or strengthened forming the best type of hypotheses for relaxing or strengthening.

Structural limitations defined by forbidden subgraphs (see (b_2) in previous section), form the next type of well defined hypotheses in view of relaxing or strengthening. Consider the following theorem based on structural limitations of this type.

Theorem C (Broersma and Veldman [14], 1997). Every 2-connected $\{K_{1,3}, P_6\}$ -free graph is hamiltonian.

Generally, it is difficult to check the sharpness related to forbidden subgraphs. However, the following result essentially simplifies this procedure in Theorem C.

Theorem D (Faudree and Gould [27], 1997). Let R and S be connected graphs ($R, S \neq P_3$) and G be a 2-connected graph of order $n \geq 10$. Then G is (R, S) -free implies G is hamiltonian if and only if $R = K_{1,3}$ and S is one of the graphs: $P_4, P_5, P_6, N_{0,0,0}, N_{0,0,1}, N_{0,0,2}, N_{0,1,1}, N_{0,0,3}, N_{0,1,2}$ or $N_{1,1,1}$.

By Theorem D, the condition " G is P_6 -free" in Theorem C can not be relaxed by replacing it with " G is H -free" for each

$$H \in \{P_4, P_5, N_{0,0,0}, N_{0,0,1}, N_{0,0,2}, N_{0,1,1}, N_{0,0,3}, N_{0,1,2}, N_{1,1,1}\}.$$

Further, the condition " G is $\{K_{1,3}, P_6\}$ -free" in Theorem C can not be relaxed by replacing it with " G is $K_{1,3}$ -free" or " G is P_6 -free" by the following theorem.

Theorem E (Faudree and Gould [27], 1997). Let R be a connected graph and G be a 2-connected graph. Then G is R -free implies G is hamiltonian if and only if $R = P_3$.

Finally, the graph $2K_6 + K_1$ shows that the condition $\kappa \geq 2$ in Theorem C can not be replaced by $\kappa \geq 1$.

So, Theorem C, as well as Theorems 21-25 are best possible.

Now consider the third type of conditions providing special graph environments (see (b3) in previous section) such as regular, bipartite, interval, chordal, line, spider, split, comparability, transitive graphs, powers of graphs and so on. If the condition cannot be gradually relaxed, it must be removed from the list of conditions to watch the behavior of the result. Clearly, r -regularity and $(r+1)$ -regularity are noncomparable and when we want to relax a condition such as " G is r -regular", we must remove this condition.

By relaxing the condition " G is bipartite" we get a trivial case when " G is one-partite" or empty graph.

Planarity can be interpreted both in view of forbidden subgraphs and embedding in a plane without crossings. The following well-known theorem is similar to Theorem G and shows that in both cases we get a non planar graph when we try to relax the planarity condition .

Theorem F (Kuratowski [44], 1930). A graph is planar if and only if it does not contain a subgraph that is homeomorphic to K_5 or $K_{3,3}$.

7 Evolution mechanisms in large cycles theory

We distinguish the following kinds of evolution mechanisms in large cycles theory:

- (c1) improvements (vertical evolution),
- (c2) modifications (horizontal evolution),
- (c3) vertical generalizations (vertical evolution leap based on inductive reasoning),
- (c4) horizontal generalizations (horizontal evolution leap based on inductive reasoning),
- (c5) involving new genetic units (genome extension).

Improvement (or strengthening) is a progressive (vertical) iterative process in evolution toward finding better results.

Definition 4. Improvement is one of the following procedures:

- (d1) relaxing one of the conditions and preserving the conclusion,

(d2) strengthening the conclusion and preserving the conditions.

Improvements are applicable to trivial (primitive) or incomplete (not sharp) results. The following propositions are trivial and need further improvements:

- (e1) every graph with $q \geq n(n-1)/2$ is hamiltonian,
- (e2) every graph with $\delta \geq n-1$ is hamiltonian.

For example, (e2) is far from to be best possible and can be improved by relaxing the condition $\delta \geq n-1$ to $\delta \geq n-2$ when $n \geq 4$. The latter is not so trivial and still need further improvements.

Modification is a horizontal developmental process in evolution generating noncomparable results.

Definition 5. Modification is one of the following procedures:

- (f1) relaxing of some conditions, at the same time strengthening some others, under the same conclusion,
- (f2) relaxing of some conditions, at the same time relaxing the conclusion,
- (f3) strengthening of some conditions, at the same time strengthening the conclusion.

Definition 6. Vertical generalization is a leap in improvement process based on inductive reasoning toward finding best possible results.

Inductive reasoning is also known as induction: a kind of reasoning that constructs or evaluates propositions that are abstractions of observations of individual instances. For example, Theorem A can be considered as a vertical generalization of (e2). During such iterative improving processes, all intermediate cases are immediately forgotten and eliminated preserving the final best possible version claiming that "every graph with $\delta \geq n/2$ is hamiltonian". In result, the evolution process seems discrete with large breaking-offs. However, many well-known theorems still are not best possible.

Definition 7. Horizontal generalization is a leap in modification process based on inductive reasoning toward finding best possible results.

The following well-known theorem is a horizontal generalization of Theorem A.

Theorem G (Ore [55], 1960). If G is a graph such that $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices x and y , then G is Hamiltonian.

Some generalized theorems such as Theorems 14, 36, 42, 43, 44, 51, 57 (see Section 9) can be considered as unions (not generalizations) of different modifications for all possible values of λ .

Incorporation of every new genetic unit in theorems becomes a turning point in evolution in the area. Every new genetic unit appears in improvement and modification processes when we try to relax one of the conditions and preserve the conclusion. At the first stages, they appear in forms of particular values and then become parameters due to vertical or horizontal generalizations.

The order n and size q as gene elements one by one are neutral graph invariants with respect to cycle structures. Meanwhile, they become more active combined together (as in Theorem 1).

The minimum degree δ plays a central role in majority of hamiltonian results. It is not too primitive and not too complicated, becoming the most flexible invariant for various possible generalizations. Minimum degree is a more essential invariant than the order and size, providing some dispersion of the edges in a graph. The combinations between order n and minimum degree become much more fruitful especially under some additional connectivity conditions.

The impact of some relations on cycle structures can be strengthened under additional conditions of the type $\delta \geq \alpha \pm i$ if for appropriate integer i . Determining the independence number α is shown in [31] to be *NP*-hard problem.

Connectivity is the most valuable research tool toward cognition of large cycle structures. In [23], it was proved that connectivity κ can be determined in polynomial time. Many graph theorists think that the connectivity is at the heart of all path and cycle questions providing comparatively more uniform dispersion of the edges.

The binding number $b(G)$ is a measure of how well-knot a graph is. Like the connectivity, the binding number also can be computed in polynomial time, using network techniques [19].

An alternate connectedness measure is toughness τ - the most powerful and less investigated graph invariant introduced by Chvátal [16] as a means of studying the cycle structure of graphs. Moreover, it was proved [2] that for any positive rational number t , recognizing t -tough graphs (in particular 1-tough graphs) is an *NP*-hard problem. Chvátal [16] conjectured that there exists a finite constant i_0 such that every i_0 -tough graph is hamiltonian. This conjecture is still open.

For a given cycle C , the idea of using $G \setminus C$ appropriate structures lies in the base of almost all existing proof techniques in trying to construct longer cycles in graphs by the following standard procedure: choose an initial cycle C_0 in G and try to enlarge it by replacing a segment P' of C_0 with a suitable path P'' longer than P' , having the same end vertices and passing through $G \setminus C_0$. To find suitable P' and P'' , one can use the paths or cycles (preferably large) in $G \setminus C_0$ and connections (preferably high) between these paths (cycles) and C_0 . The latter are closely related to $\overline{p}, \overline{c}$, as well as minimum degree δ (local connections) and connectivity κ (global connections).

Forbidden small subgraphs provide the next powerful gene element of structural nature that directly force the graph to have large cycles. For example, 2-connected P_3 -free graphs are hamiltonian since they are complete graphs. The most common of forbidden subgraphs is the claw $K_{1,3}$.

Finally, some special graph classes, that can be defined by direct description,

provide convenient environments to construct large cycles in graphs. They are regular graphs, planar graphs, bipartite graphs, chordal graphs, interval graphs and so on.

The following theorem can be considered as a modification of Theorem A obtained by relaxing the main condition $\delta \geq n/2$ in Theorem A to $\delta \geq (n+2)/3$, at the same time involving two new invariants with limitations $\kappa \geq 2$ and $\delta \geq \alpha$.

Theorem H (Nash-Williams, 1971) [46]. Every graph is hamiltonian if $\kappa \geq 2$ and $\delta \geq \max \{(n+2)/3, \alpha\}$.

Another modification of Theorem A is based on a special graph environment involved by special structural limitation (being bipartite), at the same time essentially relaxing the single condition in Theorem A to $\delta \geq \frac{1}{4}(n+2)$.

Theorem I (Moon and Moser [45], 1963). Every balanced bipartite graph is hamiltonian if $\delta \geq \frac{1}{4}(n+2)$.

8 On fundamental results in large cycles theory

What makes a theorem (problem, conjecture) beautiful? By G.H. Hardy, "The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics".

In [11], Bondy introduced some criteria to classify conjectures, which can be applicable for theorems as well:

- Simplicity: short, easily understandable statement relating basic concepts.
- Element of Surprise: links together seemingly disparate concepts.
- Generality: valid for a wide variety of objects.
- Centrality: close ties with a number of existing theorems and/or conjectures.
- Longevity: at least twenty years old.
- Fecundity: attempts to prove the conjecture have led to new concepts or new proof techniques.

However, the first formal criterion toward classification theorems and conjectures is the property to be best possible (sharp, tight), widely applicable in combinatorics and graph theory. This criterion after some improvement can be applicable in other areas of science.

In this paper we introduce the next formal criterion to distinguish some top theorems in large cycles theory called "fundamental" based on all exact branchings of the tree of generalizations. By this approach, all results in large cycles

theory have descended from a number of common ancestors (fundamental result) through generalizations. Fundamental results can not be directly improved and can be evolved only by modifications and generalizations.

The term "fundamental result" is used in various fields of science to characterize mainly the central and most important results in the area, based on subjective perception. In this paper, this term is used according to the second much more important mean: "forming the source or base from which everything else is made; not able to be divided any further". Observe also that in general, there are no physical and abstract units in the nature, lying in the base of all material or abstract notions. However, every notion in large cycles theory has certain origins due to certain frames of this theory.

9 The list of fundamental results

9.1 Hamilton cycles

Theorem 1 (Erdős and Gallai, 1959) [25]

Every graph is hamiltonian if

$$q \geq \frac{n^2 - 3n + 5}{2}.$$

Example for sharpness. To see that the size bound $(n^2 - 3n + 5)/2$ in Theorem 1 is best possible, note that the graph formed by joining one vertex of K_{n-1} to K_1 , contains $(n^2 - 3n + 4)/2$ edges and is not hamiltonian.

Theorem 2 (Erdős, 1962) [24]

Every graph is hamiltonian if $1 \leq \delta \leq n/2$ and

$$q > \max \left\{ \frac{(n - \delta)(n - \delta - 1)}{2} + \delta^2, \frac{(n - \lfloor \frac{n-1}{2} \rfloor)(n - \lfloor \frac{n-1}{2} \rfloor - 1)}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2 \right\}.$$

Example for sharpness. The graph consisting of a complete graph on $n - \delta$ vertices, δ of which are joined to each of δ independent vertices, shows that the condition in Theorem 2 cannot be weakened.

Theorem 3 (Moon and Moser, 1963) [45]

Every balanced bipartite graph is hamiltonian if

$$q \geq \frac{n^2 - 2n + 5}{4}.$$

Examples for sharpness. Clearly, the condition " G is balanced" in Theorem 3 can not be removed. The graph obtained from $K_{t,t}$ by deleting $t - 1$ edges with a common vertex, shows that the condition $q \geq (n^2 - 2n + 5)/4$ in Theorem 3 can not be replaced by $q \geq (n^2 - 2n + 4)/4$.

Theorem 4 (Moon and Moser, 1963) [45]
Every balanced bipartite graph is hamiltonian if

$$q > \frac{n(n-2\delta)}{4} + \delta^2.$$

Examples for sharpness. Clearly, the condition "G is balanced" in Theorem 4 can not be removed. Consider the balanced bipartite graph $G = (X, Y; E)$ with vertex classes of the form $X = P \cup Q$, $Y = R \cup S$, where $|P| = |R| = \delta$, $|Q| = |S| = n/2 - \delta$, $N_G(x) = R$ for all $x \in P$, and $N_G(x) = Y$ for all $x \in Q$. This example shows that Theorem 4 is best possible.

Theorem 5 (Nikoghosyan, 2011) [51]
Every graph is hamiltonian if

$$q \leq \delta^2 + \delta - 1.$$

Example for sharpness. $K_1 + 2K_\delta$.

Theorem 6 (Dirac, 1952) [21]
Every graph is hamiltonian if

$$\delta \geq \frac{n}{2}.$$

Example for sharpness. $2K_\delta + K_1$.

Theorem 7 (Moon and Moser, 1963) [45]
Every balanced bipartite graph is hamiltonian if

$$\delta \geq \frac{n+1}{4}.$$

Examples for sharpness. Clearly, the condition "G is balanced" in Theorem 7 can not be removed. Since n is even, the condition $\delta \geq (n+1)/4$ in Theorem 7 yields a stronger condition $\delta \geq (n+2)/4$. Let $P_i = x_i y_i z_i w_i$ ($i = 1, 2, 3$) be three disjoint paths. Form a graph from P_1, P_2, P_3 by identifying x_1, x_2, x_3 in one vertex and w_1, w_2, w_3 in another vertex. The resulting graph shows that the condition $\delta \geq (n+1)/4$ in Theorem 7 can not be replaced by $\delta \geq n/4$.

Theorem 8 (Jung, 1978) [35]
Every graph is hamiltonian if $n \geq 11$, $\tau \geq 1$ and

$$\delta \geq \frac{n-4}{2}.$$

Examples for sharpness. Petersen graph; $K_{\delta, \delta+1}$; G_n^* .

Theorem 9 (Nikoghosyan, 2012) [53]
Every graph is hamiltonian if $\tau > 4/3$ and

$$\delta \geq \frac{n-5}{2}.$$

Examples for sharpness. The Petersen graph shows that the condition $\tau > 4/3$ in Theorem 9 can not be replaced by $\tau = 4/3$. Let H_1 be a complete graph with vertex set $V(H_1) = \{x_1, x_2, x_3, x_4, x_5\}$ and H_2 a complete bipartite graph with bipartition (V_1, V_2) , where $V_1 = \{y_1, y_2, y_3, y_4, y_5\}$ and $|V_2| = 2$. The graph obtained from disjoint graphs H_1 and H_2 by adding the edges $x_i y_i$ ($i = 1, 2, 3, 4, 5$), shows that the condition $\delta \geq (n - 5)/2$ in Theorem 9 can not be replaced by $\delta \geq (n - 6)/2$.

Theorem 10 (Nikoghosyan, 1981) [48]
Every graph is hamiltonian if $\kappa \geq 2$ and

$$\delta \geq \frac{n + \kappa}{3}.$$

Examples for sharpness. $2K_\delta + K_1$; $H(1, \delta - \kappa + 1, \delta, \kappa)$ ($2 \leq \kappa < n/2$).

Theorem 11 (Bauer and Schmeichel, 1991) [5]
Every graph is hamiltonian if $\tau \geq 1$ and

$$\delta \geq \frac{n + \kappa - 2}{3}.$$

Examples for sharpness. $K_{\delta, \delta+1}$; L_δ .

Theorem 12 (Nash-Williams, 1971) [46]
Every graph is hamiltonian if $\kappa \geq 2$ and

$$\delta \geq \max \left\{ \frac{n + 2}{3}, \alpha \right\}.$$

Examples for sharpness. $(\lambda + 1)K_{\delta-\lambda+1} + K_\lambda$ ($\delta \geq 2\lambda$); $(\lambda + 2)K_{\delta-\lambda} + K_{\lambda+1}$ ($\delta \geq 2\lambda + 1$); $H(\lambda, \lambda + 1, \lambda + 3, \lambda + 2)$.

Theorem 13 (Bigalke and Jung, 1979) [9]
Every graph is hamiltonian if $\tau \geq 1$ and

$$\delta \geq \max \left\{ \frac{n}{3}, \alpha - 1 \right\}.$$

Examples for sharpness. $K_{\delta, \delta+1}$ ($n \geq 3$); L_δ ($n \geq 7$); $K_{\delta, \delta+1}$ ($n \geq 3$).

Theorem 14 (Fraisse, 1986) [30]
Let G be a graph and λ a positive integer. Then G is hamiltonian if $\kappa \geq \lambda + 1$ and

$$\delta \geq \max \left\{ \frac{n + 2}{\lambda + 2} + \lambda - 1, \alpha + \lambda - 1 \right\}.$$

Examples for sharpness. $(\lambda + 1)K_{\delta-\lambda+1} + K_\lambda$ ($\delta \geq 2\lambda$); $(\lambda + 2)K_{\delta-\lambda} + K_{\lambda+1}$ ($\delta \geq 2\lambda + 1$); $H(\lambda, \lambda + 1, \lambda + 3, \lambda + 2)$. Theorem 14 can be considered as a union

(not a generalization) of fundamental results for all possible values of λ .

Theorem 15 (Yamashita, 2008) [60]

Every graph is hamiltonian if $\kappa \geq 3$ and

$$\delta \geq \max \left\{ \frac{n + \kappa + 3}{4}, \alpha \right\}.$$

Examples for sharpness. $3K_{\delta-1} + K_2$; $H(2, n-3\delta+3, \delta-1, \kappa)$; $H(1, 2, \kappa+1, \kappa)$.

Theorem 16 (Chvátal and Erdős, 1972) [17]

Every graph is hamiltonian if

$$\kappa \geq \alpha.$$

Example for sharpness. $K_{\delta, \delta+1}$.

Theorem 17 (Woodall, 1973) [58]

Every graph G is hamiltonian if

$$b(G) \geq \frac{3}{2}.$$

Example for sharpness. $aK_2 + \overline{K}_{a-1}$.

Theorem 18 (Fleischner, 1974) [29]

The square of every 2-connected graph is hamiltonian.

Examples for sharpness. Clearly, the power of a graph can not be reduced to one in Theorem 18, since there are 2-connected nonhamiltonian graphs. Next, 2-connectivity condition in Theorem 18 can not be relaxed since the square of a graph G is not hamiltonian if $G - x$ has at least three nontrivial components in which x has exactly one neighbor.

Theorem 19 (Tutte, 1956) [57]

Every 4-connected planar graph is hamiltonian.

Examples for sharpness. Tutte's graph shows that 4-connectivity condition in Theorem 19 can not be relaxed. Complete bipartite graph $K_{4,5}$ shows that planarity is a necessary condition in Theorem 19.

Theorem 20 (R. Thomas and X. Yu, 1994) [56]

Every 4-connected projective-plane graph is hamiltonian.

Examples for sharpness. The simplest non-orientable surface on which the Petersen graph can be embedded without crossings is the projective plane. The Petersen graph shows that 4-connectivity condition in Theorem 20 can not be relaxed. On the other hand, there are 4-connected non hamiltonian graphs that

can not be embedded on projective plane (otherwise, all 4-connected graphs are hamiltonian), implying that the condition "G is projective plane graph" can not be removed in Theorem 20.

Theorem 21 (Faudree and Gould, 1997) [27]
Every 2-connected P_3 -free graph is hamiltonian.

Examples for sharpness. See Subsection 5.3.

Theorem 22 (Broersma, Veldman, 1997) [14]
Every 2-connected $\{K_{1,3}, P_6\}$ -free graph is hamiltonian.

Examples for sharpness. See Subsection 5.3.

Theorem 23 (Faudree, Gould, Ryjáček and Schiermeyer, 1997) [28]
Every 2-connected $\{K_{1,3}, N_{0,0,3}\}$ -free graph with $n \geq 10$ is hamiltonian.

Examples for sharpness. See Subsection 5.3.

Theorem 24 (Bedrossian, 1997) [7]
Every 2-connected $\{K_{1,3}, N_{0,1,2}\}$ -free graph is hamiltonian.

Examples for sharpness. See Subsection 5.3.

Theorem 25 (Duffus, Jakobson and Gould, 1997) [22]
Every 2-connected $\{K_{1,3}, N_{1,1,1}\}$ -free graph is hamiltonian.

Examples for sharpness. See Subsection 5.3.

Theorem 26 (Keil, 1985) [41]
Every 1-tough interval graph is hamiltonian.

Examples for sharpness. Star graphs are interval nonhamiltonian graphs with $\tau < 1$, implying that 1-toughness condition in Theorem 26 can not be relaxed. The Petersen graph shows that the condition "G is interval graph" in Theorem 26 can not be removed.

Theorem 27 (Kratsch, Lehel and Müller, 1996) [43]
Every $3/2$ -tough split graph is hamiltonian.

Examples for sharpness. In [43], $(3/2 - \epsilon)$ -tough split graphs are constructed that are not hamiltonian. There are non hamiltonian graphs with $\tau = 9/4 - \epsilon > 3/2$, implying that the condition "G is split graph" in Theorem 27 can not be removed.

Theorem 28 (Deogun, Kratsch and Steiner, 1997) [20]

Every 1-tough cocomparability graph is hamiltonian.

Examples for sharpness. Clearly, any complete graph is a comparability graph and hence, any empty graph is a cocomparability graph with $\tau < 1$, implying that the condition "G is 1-tough" in Theorem 28 can not be relaxed. On the other hand, there are 1-tough non hamiltonian non cocomparability graphs (otherwise, all 1-tough graphs are hamiltonian), implying that the condition "G is cocomparability graph" in Theorem 28 can not be removed.

Theorem 29 (Böhme, Harant and Tkáč, 1999) [10]
Every chordal, planar graph with $\tau > 1$ is hamiltonian.

Examples for sharpness. In [10], it is proved that for any $\epsilon > 0$, there is a 1-tough chordal planar graph G_ϵ such that the length of a longest cycle of G_ϵ is less than $\epsilon|V(G_\epsilon)|$, implying that the condition $\tau > 1$ in Theorem 29 can not be relaxed. Chvátal [16] obtained $(3/2 - \epsilon)$ -tough graphs without a 2-factor, implying that the planarity condition in Theorem 29 can not be removed. Finally, Harant [34] found $3/2$ -tough planar nonhamiltonian graphs, implying that the condition "G is chordal" in Theorem 29 can not be removed.

Theorem 30 (Kaiser, Král and Stacho, 2007) [38]
Every $3/2$ -tough spider (intersection) graph is hamiltonian.

Examples for sharpness. In [38], Kaiser, Král and Stacho constructed $(3/2 - \epsilon)$ -tough spider graphs that do not contain a Hamilton cycle, implying that the condition "G is $3/2$ -tough" in Theorem 30 can not be relaxed. On the other hand, the condition "G is spider graph" in Theorem 30 can not be removed since there are $3/2$ -tough nonhamiltonian graphs.

9.2 Dominating cycles

Theorem 31 (Nikoghosyan, 2011) [52]
Let G be a graph. Then each longest cycle in G is a dominating cycle if $\kappa \geq 2$ and

$$q \leq \begin{cases} 8 & \text{if } \delta = 2, \\ \frac{3(\delta-1)(\delta+2)-1}{2} & \text{if } \delta \geq 3. \end{cases}$$

Examples for sharpness. To show that Theorem 31 is sharp, suppose first that $\delta = 2$. The graph $K_1 + 2K_2$ shows that the connectivity condition $\kappa \geq 2$ in Theorem 31 can not be relaxed by replacing it with $\kappa \geq 1$. The graph with vertex set $\{v_1, v_2, \dots, v_8\}$ and edge set

$$\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1v_7, v_7v_8, v_8v_4\},$$

shows that the size bound $q \leq 8$ can not be relaxed by replacing it with $q \leq 9$. Finally, the graph $K_2 + 3K_1$ shows that the conclusion "each longest cycle in G

is a dominating cycle” can not be strengthened by replacing it with ” G is hamiltonian”. Analogously, we can use $K_1 + 2K_\delta$, $K_2 + 3K_{\delta-1}$ and $K_\delta + (\delta + 1)K_1$, respectively, to show that Theorem 31 is sharp when $\delta \geq 3$. So, Theorem 31 is best possible in all respects.

Theorem 32 (Nash-Williams, 1971) [46]

Let G be a graph. Then each longest cycle in G is a dominating cycle if $\kappa \geq 2$ and

$$\delta \geq \frac{n+2}{3}.$$

Examples for sharpness. $2K_3 + K_1$; $3K_{\delta-1} + K_2$; $H(1, 2, 4, 3)$.

The graph $2K_3 + K_1$ shows that the connectivity condition $\kappa \geq 2$ in Theorem 32 can not be replaced by $\kappa \geq 1$. The second graph shows that the minimum degree condition $\delta \geq (n+2)/3$ can not be replaced by $\delta \geq (n+1)/2$. Finally, the third graph shows that the conclusion ”is a dominating cycle” can not be strengthened by replacing it with ”is a Hamilton cycle”.

Theorem 33 (Bigalke and Jung, 1979) [9]

Let G be a graph. Then each longest cycle in G is a dominating cycle if $\tau \geq 1$ and

$$\delta \geq \frac{n}{3}.$$

Examples for sharpness. $2(\kappa+1)K_2 + \kappa K_1$; L_3 ; G_n^* .

Theorem 34 (Yamashita, 2008) [60]

Let G be graph. Then each longest cycle in G is a dominating cycle if $\kappa \geq 3$ and

$$\delta \geq \frac{n+\kappa+3}{4}.$$

Examples for sharpness. $3K_{\delta-1} + K_2$; $H(2, n-3\delta+3, \delta-1, \kappa)$; $H(1, 2, \kappa+1, \kappa)$.

9.3 CD_λ -cycles

Theorem 35 (Jung, 1990) [37]

Let G be a graph. Then each longest cycle in G is a CD_3 -cycle if $\kappa \geq 3$ and

$$\delta \geq \frac{n+6}{4}.$$

Examples for sharpness. $\lambda K_{\lambda+1} + K_{\lambda-1}$ ($\lambda \geq 2$) ; $(\lambda+1)K_{\delta-\lambda+1} + K_\lambda$ ($\lambda \geq 1$) ; $H(\lambda-1, \lambda, \lambda+2, \lambda+1)$ ($\lambda \geq 2$).

Theorem 36 (Nikoghosyan, 2009) [50]

Let G be a graph and λ a positive integer. Then each longest cycle in G is a $CD_{\min\{\lambda, \delta-\lambda+1\}}$ -cycle if $\kappa \geq \lambda$ and

$$\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2.$$

Examples for sharpness. $\lambda K_{\lambda+1} + K_{\lambda-1}$ ($\lambda \geq 2$) ; $(\lambda+1)K_{\delta-\lambda+1} + K_{\lambda}$ ($\lambda \geq 1$) ; $H(\lambda-1, \lambda, \lambda+2, \lambda+1)$ ($\lambda \geq 2$).

9.4 Long cycles

Theorem 37 (Dirac, 1952) [21]

In every graph,

$$c \geq \delta + 1.$$

Example for sharpness. Join two copies of $K_{\delta+1}$ by an edge.

Theorem 38 (Kouider, 1994) [42]

In every graph,

$$c \geq \frac{n}{\lceil \alpha/\kappa \rceil}.$$

Example for sharpness. Complete bipartite graph with $\kappa = \alpha$ shows that the bound in Theorem 38 is sharp. The original result is formulated for 2-connected graphs. However, Theorem 38 is true under assumption that each vertex (edge) is a cycle of length one (two, respectively).

Theorem 39 (Nikoghosyan, 1998) [54]

Let G be a graph and C a longest cycle in G . Then

$$|C| \geq (\bar{p} + 2)(\delta - \bar{p}).$$

Example for sharpness. $(\kappa + 1)K_{\delta-\kappa+1} + K_{\kappa}$.

Theorem 40 (Nikoghosyan, 2000) [54]

Let G be a graph and C a longest cycle in G . Then

$$|C| \geq (\bar{e} + 1)(\delta - \bar{e} + 1).$$

Example for sharpness. $(\kappa + 1)K_{\delta-\kappa+1} + K_{\kappa}$.

Theorem 41 (Nikoghosyan, 2000) [49]

Let G be a graph with $\kappa \geq 2$ and C a longest cycle in G . If $\bar{e} \geq \kappa$ then

$$|C| \geq \frac{(\bar{e} + 1)\kappa}{\bar{e} + \kappa + 1}(\delta + 2).$$

Otherwise,

$$|C| \geq \frac{(\bar{e} + 1)\bar{e}}{2\bar{e} + 1}(\delta + 2).$$

Example for sharpness. $(\kappa + 1)K_{\delta-\kappa+1} + K_{\kappa}$.

9.5 Hamilton cycles and long cycles

Theorem 42 (Woodall, 1976) [59]

Let G be a graph and λ, t, r be integers with $n = t(\lambda - 1) + r + 1$, where $\lambda \geq 2$, $t \geq 0$ and $0 \leq r < \lambda - 1$. If

$$q > t \binom{\lambda}{2} + \binom{r+1}{2}$$

then

$$c > \lambda.$$

Example for sharpness. The result is best possible, in view of the graph consisting of t copies of K_λ and one copy of K_{r+1} , all having exactly one vertex in common.

Theorem 43 (Fan, Lv and Wang, 2004) [26]

Let G be a 2-connected graph and let $2 \leq \lambda \leq n - 1$. If

$$q > \max \left\{ f(n, 2, \lambda), f(n, \left\lfloor \frac{\lambda}{2} \right\rfloor, \lambda) \right\}$$

then

$$c > \lambda,$$

where $f(n, t, \lambda) = (\lambda + 1 - t)(\lambda - t)/2 + t(n - \lambda - 1 + t)$ and $2 \leq t \leq \lambda/2$.

Examples for sharpness. The result is best possible, in view of the graph obtained from $K_{\lambda+1-t}$ by adding $n - (\lambda + 1 - t)$ isolated vertices, each joined to the same t vertices of $K_{\lambda+1-t}$.

Theorem 44 (Alon, 1986) [1]

Let G be a graph and λ a positive integer. If $\delta \geq \frac{n}{\lambda+1}$ then

$$c \geq \frac{n}{\lambda}.$$

Examples for sharpness. $(\lambda + 1)K_\lambda + K_1$; $\lambda K_{\lambda+1}$.

Theorem 45 (Dirac, 1952) [21]

Let G be a graph. If $\kappa \geq 2$ then

$$c \geq \min\{n, 2\delta\}.$$

Examples for sharpness. $(\lambda + 1)K_{\lambda+1} + K_\lambda$ ($\lambda \geq 1$); $(\lambda + 3)K_{\lambda-1} + K_{\lambda+2}$ ($\lambda \geq 2$); $(\lambda + 2)K_\lambda + K_{\lambda+1}$ ($\lambda \geq 1$).

Theorem 46 (Kaneko and Yoshimoto, 1952) [39]

Let G be a 2-connected balanced bipartite graph. Then

$$c \geq \min\{n, 4\delta - 2\}.$$

Examples for sharpness. Clearly, the condition " G is balanced" in Theorem 46 can not be removed. Consider the balanced bipartite graph $G = (X, Y; E)$ with vertex classes of the form $X = P \cup Q$, $Y = R \cup S$ with $z \in Q$, where $|P| = |R| = |Q| = |S| = n/4$, $N_G(x) = R$ for all $x \in P$, $N_G(x) = S$ for all $x \in Q - z$ and $N_G(z) = Y$. This example shows that 2-connectivity condition in Theorem 46 cannot be weakened. Next, consider the balanced bipartite graph $G = (X, Y; E)$ with vertex classes of the form $X = P \cup Q$, $Y = R \cup S$, where $|P| = |R| = |Q| = |S| = n/4$, $N_G(x) = R$ for all $x \in P$, and $N_G(x) = Y$ for all $x \in Q$. This example shows that the bound $4\delta - 2$ in Theorem 46 cannot be improved.

Theorem 47 (Bauer and Schmeichel, 1987) [4]

Let G be a graph. If $\tau \geq 1$ then

$$c \geq \min\{n, 2\delta + 2\}.$$

Examples for sharpness. $K_{\delta, \delta+1}$; L_2 .

Theorem 48 (Nikoghosyan, 2012) [53]

Let G be a graph. If $\tau > 4/3$ then

$$c \geq \min\{n, 2\delta + 5\}.$$

Examples for sharpness. The Petersen graph shows that the condition $\tau > 4/3$ in Theorem 48 cannot be replaced by $\tau = 4/3$. Let H_1 be a complete bipartite graph with bipartition $V_1 = \{x_1, x_2, x_3, x_4, x_5\}$ and $V_2 = \{y_1, y_2\}$, and let H_2 be a complete graph with vertex set $V = \{z_1, z_2, z_3, z_4, z_5\}$. The graph obtained from disjoint graphs H_1 and H_2 by adding the edges $x_i z_i$ ($i = 1, \dots, 5$), shows that the bound $c \geq 2\delta + 5$ in Theorem 48 can not be replaced by $c \geq 2\delta + 6$.

Theorem 49 (Nikoghosyan, 1981) [48]

Let G be a graph. If $\kappa \geq 3$ then

$$c \geq \min\{n, 3\delta - \kappa\}.$$

Examples for sharpness. $3K_{\delta-1} + K_2$; $H(1, \delta - \kappa + 1, \delta, \kappa)$.

Theorem 50 (Jung, 1978) [35]

Let G be a graph. If $\kappa \geq 3$ and $\delta \geq \alpha$ then

$$c \geq \min\{n, 3\delta - 3\}.$$

Examples for sharpness. $(\lambda + 2)K_{\lambda+2} + K_{\lambda+1}$; $(\lambda + 4)K_\lambda + K_{\lambda+3}$; $(\lambda + 3)K_{\lambda+1} + K_{\lambda+2}$.

Theorem 51 (Nikoghosyan, 2009) [50]

Let G be a graph and λ a positive integer. If $\kappa \geq \lambda + 2$ and $\delta \geq \alpha + \lambda - 1$ then

$$c \geq \min\{n, (\lambda + 2)(\delta - \lambda)\}.$$

Examples for sharpness. $(\lambda + 2)K_{\lambda+2} + K_{\lambda+1}$; $(\lambda + 4)K_{\lambda} + K_{\lambda+3}$; $(\lambda + 3)K_{\lambda+1} + K_{\lambda+2}$.

Theorem 52 (M.Zh. Nikoghosyan and Zh.G. Nikoghosyan, 2011) [47]
Let G be a graph. If $\kappa \geq 4$ and $\delta \geq \alpha$ then

$$c \geq \min\{n, 4\delta - \kappa - 4\}.$$

Examples for sharpness. $4K_{\delta-2} + K_3$; $H(1, 2, \kappa+1, \kappa)$; $H(2, n-3\delta+3, \delta-1, \kappa)$.

Theorem 53 (Bauer, Morgana, Schmeichel and Veldman, 1989) [3]
Let G be a graph. If $\kappa \geq 2$ and $\delta \geq \frac{n+2}{3}$ then

$$c \geq \min\{n, n + \delta - \alpha\}.$$

Examples for sharpness. $2K_{\delta} + K_1$; $3K_{\delta-1} + K_2$; $K_{2\delta-2, \delta}$.

Theorem 54 (Bauer, Schmeichel and Veldman, 1988) [6]
Let G be a graph. If $\tau \geq 1$ and $\delta \geq \frac{n}{3}$ then

$$c \geq \min\{n, n + \delta - \alpha + 1\}.$$

Examples for sharpness. $K_{\delta, \delta+1}$; L_{δ} ; G_n^* .

9.6 Dominating cycles and long cycles

Theorem 55 (Jung, 1981) [36]

Let G be a graph. If $\kappa \geq 3$ then either each longest cycle in G is a dominating cycle or

$$c \geq 3\delta - 3.$$

Examples for sharpness. $(\lambda + 1)K_{\lambda+1} + K_{\lambda}$ ($\lambda \geq 1$); $(\lambda + 3)K_{\lambda-1} + K_{\lambda+2}$ ($\lambda \geq 2$); $(\lambda + 2)K_{\lambda} + K_{\lambda+1}$ ($\lambda \geq 1$).

Theorem 56 (M.Zh. Nikoghosyan and Zh.G. Nikoghosyan, 2011) [47]

Let G be a graph. If $\kappa \geq 4$ then either each longest cycle in G is a dominating cycle or

$$c \geq 4\delta - \kappa - 4.$$

Examples for sharpness. $4K_{\delta-2} + K_3$; $H(2, \delta - \kappa + 1, \delta - 1, \kappa)$; $H(1, 2, \kappa + 1, \kappa)$.

9.7 CD_{λ} -cycles and long cycles

Theorem 57 (Nikoghosyan, 2009) [50]

Let G be a graph and λ a positive integer. If $\kappa \geq \lambda + 1$ then either each longest cycle in G is a $CD_{\min\{\lambda, \delta - \lambda\}}$ -cycle or

$$c \geq (\lambda + 1)(\delta - \lambda + 1).$$

Examples for sharpness. $(\lambda + 1)K_{\lambda+1} + K_{\lambda}$ ($\lambda \geq 1$); $(\lambda + 3)K_{\lambda-1} + K_{\lambda+2}$ ($\lambda \geq 2$); $(\lambda + 2)K_{\lambda} + K_{\lambda+1}$ ($\lambda \geq 1$).

References

- [1] N. Alon, The longest cycle of a graph with a large minimum degree, *J. Graph Theory* 10 (1986) 123-127.
- [2] D. Bauer, S.L. Hakimi and E. Schmeichel, Recognizing tough graphs is *NP*-hard, *Discrete Appl. Math.* 28 (1990) 191-195.
- [3] D. Bauer, A. Morgana, E. Schmeichel and H.J. Veldman, Long cycles in graphs with large degree sums, *Discrete Math.* 79 (1989/90) 59-70.
- [4] D. Bauer and E. Schmeichel, Long cycles in tough graphs, preprint (1987).
- [5] D. Bauer and E. Schmeichel, On a theorem of Häggkvist and Nicoghossian, *Graph Theory, Combinatorics, Algorithms and Applications* (1991) 20-25.
- [6] D. Bauer, E. Schmeichel and H.J. Veldman, A generalization of a theorem of Bigalke and Jung, *Ars Combinatoria*, v.26 (1988) 53-58.
- [7] P. Bedrossian, Forbidden subgraphs and minimum degree conditions for hamiltonicity, PhD thesis, Memphis State University, 1991.
- [8] J.C. Bermond, Hamiltonian graphs, In: Beineke and Wilson, *Selected topics in graph theory*, Academic press, London (1978).
- [9] A. Bigalke and H.A. Jung, Über Hamiltonische Kreise und unabhängige Ecken in Graphen, *Monatsh. Math.* 88 (1979) 195-210.
- [10] T. Böhme, J. Harant and M. Tkáč, More than one tough chordal planar graphs are hamiltonian, *J. Graph Theory* 32 (1999) 405-410.
- [11] J.A. Bondy, Beautiful Conjectures in Graph Theory, available at: <http://www.ime.usp.br/pf/grafos-exercicios/bondy-conjectures/beautiful.pdf>.
- [12] J.A. Bondy, Integrity in graph theory, in: G. Chartrand, Y. Alavi, D.L. Goldsmith, L. Lesniak-Foster, D.R. Lick (Eds.), *In the Theory and Application of Graphs*, Wiley, New York, 1981, pp. 117-125. MR83e:05070.
- [13] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York (1976).
- [14] H. Broersma and H.J. Veldman, Restrictions on induced subgraphs ensuring Hamiltonicity of $K_{1,3}$ -free graphs, in: *Contemporary methods in graph theory*, Bibliographisches Inst., Mannheim (1990) 181-194.

- [15] D.T. Campbell, Blind variation and selective retention in creative thought as in other knowledge processes, *Psychological Review*, 67(6) (1960) 380-400.
- [16] V. Chvátal, Tough graphs and Hamiltonian circuits, *Discrete Math.* 5 (1973) 215-228.
- [17] V. Chvátal and P. Erdős, A note on hamiltonian circuits, *Discrete Math.* 2 (1972) 111-113.
- [18] S.A. Cook, The Complexity of Theorem-Proving Procedures, *Proceedings, Third Annual ACM Symposium on the theory of computing*, ACM, New York (1971) 151-158.
- [19] W.H. Cunningham, Computing the binding number of a graph, *Discrete Math.* 80 (1990) 283-285.
- [20] J.S. Deogun, D. Kratsch and G. Steiner, 1-Tough cocomparability graphs are hamiltonian, *Discrete Math.* 170 (1997) 99-106.
- [21] G.A. Dirac, Some theorems on abstract graphs, *Proc. London, Math. Soc.* 2 (1952) 69-81.
- [22] D. Duffus, M.S. Jacobson and R.J. Gould, Forbidden subgraphs and the Hamiltonian theme, in: *The theory and applications of graphs (Kalamazoo, Mich., 1980)* 297-316, Wiley, New York, 1981.
- [23] S. Even and R.E. Tarjan, Network flow and testing graph connectivity, *SIAM journal of computing*, 4 (1975) 507-518.
- [24] P. Erdős, Remarks on a paper of Pösa, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 7 (1962) 227-229.
- [25] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.* 10 (1959) 337-356.
- [26] G. Fan, X. Lv and P. Wang, Cycles in 2-connected graphs, *J. of Combin. Theory ser. B*92 (2004) 379-394.
- [27] R.J. Faudree and R.J. Gould, Characterizing forbidden pairs for Hamilton properties, *Discrete Math.* 173 (1997) 45-60.
- [28] R. Faudree, R. Gould, Z. Ryjáček and I. Schiermeyer, Forbidden subgraphs and pancyclicity, in: *Proceedings of the twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995)* v. 109 (1995) 13-32.
- [29] H. Fleischner, The square of every two-connected graph is hamiltonian, *J. Combin. Theory Ser B*16 (1974) 29-34.

- [30] P. Fraisse, D_λ -cycles and their applications for Hamiltonian graphs, Université de Paris-sud, preprint (1986).
- [31] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP -Completeness. New York: W.H. Freeman, 1983.
- [32] R.J. Gould, Updating the Hamiltonian Problem - A survey, J. Graph Theory 15 (1991) 121-157.
- [33] R.J. Gould, Advances on the Hamiltonian Problem - A survey, Graphs and Combinatorics 19 (2003) 7-52.
- [34] J. Harant, Toughness and nonhamiltonicity of polyhedral graphs, Discrete Math. 113 (1993) 249-253.
- [35] H.A. Jung, On maximal cycles in finite graphs, Annals of Discrete Math. 3 (1978) 129-144.
- [36] H.A. Jung, Longest circuits in 3-connected graphs, Coll. Math. Soc. J. Bolyai 37, Finite and infinite sets, Eger (1981) 403-438.
- [37] H.A. Jung, Long Cycles in Graphs with Moderate Connectivity, Topics in combinatorics and graph theory, R.Bodendieck and R.Henn (Editors), Phisika Verlag, Heidelberg (1990) 765-778.
- [38] T. Kaiser, D. Král and L. Stacho, Tough spiders, J. Graph Theory 56 (2007) 23-40.
- [39] A. Kaneko and K. Yoshimoto, On longest cycles in a 2-connected bipartite graph with Ore type condition, I (preprint) 2002.
- [40] R.M. Karp, Reducibility Among Computational Problems, in R.E. Miller and J.W. Thatcher (editors), Complexity of Computer Computations, Plenum (1972) 85-103.
- [41] J.M. Keil, Finding hamiltonian circuits in interval graphs, Inf. Proc. Let. 20 (1985) 201-206.
- [42] M. Kouider, Cycles in graphs with prescribed stability number and connectivity, J. Combin. Theory, Ser. B 60 (1994) 315-318.
- [43] D. Kratsch, J. Lehel and H. Müller, Toughness, hamiltonicity and split graphs, Discrete Math. 150 (1996) 231-245.
- [44] K. Kuratowski, Sur le problème des courbes gauches en topologie, Fund. Math. 15 (1930) 271-283.
- [45] J.W. Moon and L. Moser, On hamiltonian bipartite graphs, Israel J. Math., 1 (1963) 163-165.

- [46] C.St.J.A. Nash-Williams, Edge-disjoint hamiltonian cycles in graphs with vertices of large valency, in: L. Mirsky, ed., "Studies in Pure Mathematics", pp. 157-183, Academic Press, San Diego/London (1971).
- [47] M.Zh. Nikoghosyan and Zh.G. Nikoghosyan, Large cycles in 4-connected graphs, Discrete Math. 311 (2011) 302-306.
- [48] Zh.G. Nikoghosyan, On maximal cycle of a graph, DAN Arm. SSR v. LXXII 2 (1981) 82-87 (in Russian).
- [49] Zh.G. Nikoghosyan, Cycle-Extensions and Long Cycles in k-connected Graphs, Transactions of the Institute for Informatics and Automation Problems of the NAS (Republic of Armenia) and Yerevan State University, Mathematical Problems of Computer Science 21 (2000) 129-155.
- [50] Zh.G. Nikoghosyan, Dirac-type generalizations concerning large cycles in graphs, Discrete Math. 309 (2009) 1925-1930.
- [51] Zh.G. Nikoghosyan, A size bound for Hamilton cycles, Preprint available at <http://arxiv.org/abs/1107.2201v1> [math.CO] 12 Jul 2011.
- [52] Zh.G. Nikoghosyan, A size upper bound for dominating cycles, Preprint available at <http://arxiv.org/abs/1112.2467v1> [math.CO] 12 Dec 2011.
- [53] Zh.G. Nikoghosyan, A sharp bound for the Circumference in t -tough graphs with $t > 1$, Preprint available at <http://arxiv.org/abs/1204.6515v1> [math.CO] 29 Apr 2012.
- [54] Zh.G. Nikoghosyan, Advanced Lower Bounds for the Circumference, Graphs and Combinatorics, DOI: 10.1007/s00373-012-1209-4.
- [55] O. Ore, A note on hamiltonian circuits, Am. Math. Month. 67 (1960) 55.
- [56] R. Thomas and X. Yu, 4-connected projective-plane graphs are hamiltonian, J. Combin. Theory Ser. B62 (1994) 114-132.
- [57] W.T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956) 99-116.
- [58] D.R. Woodall, The binding number of a graph and its Anderson numbers, J. Combin. Theory Ser. b15 (1973) 225-255.
- [59] D.R. Woodall, Maximal circuits of graphs, I, Acta Math. Acad. Sci. Hungar. 28 (1976) 77-80.
- [60] T. Yamashita, A degree sum condition with connectivity for relative length of longest paths and cycles, Discrete Math. 309 (23-24) (2009) 6503-6507.

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